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The graded rings associated to cusp singularities

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Introduction

In 1983, Tsuchihashi [T] constructed an isolated singularity called a cusp singularity from a pair of an open cone and a discrete group acting on it, which is a generalization of the Hilbert modular cusp singularities. By the construction, Tsuchihashi's cusp singularity is described by the fan decomposing the open cone. Although the cusp does not depend on the choice of the fan, it describes a desingularlization of the singularity. Furthermore, some important invariants are calculated by the fan.

In this note, we will show the possibility of using computer for analyzing cusp singularities by introducing the method of Gröbner basis. Since the completed local ring of a cusp singularity is a subring of the completion of a semigroup ring, it is possible to define the leading monomials of the elements if we fix a monomial order on the semigroup. However, we have difficulties since the semigroup consisting of the leading monomials is not finitely generated in general. This indicates that monomial orders are too strong for our purpose.

In most of monomial orders, the grading is the first step of the ordering. We will show that the grading of the semigroup ring gives a filtration of the local ring, and the associated graded ring is finitely generated. Using this result, we can show that the local ring is Noetherian over a general field. In the last section, we will give some remarks on the local ring related to the theory of Gröbner basis.

1 Subrings of a semigroup ring

Let N be a free $\mathbf{Z}$-module of rank $r \geq 0$. The dual $\mathbf{Z}$-module $\mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ is denoted by M. We set $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$. N and M are contained in $N_{\mathbf{R}}$ and $M_{\mathbf{R}}$, respectively, as lattices. The natural pairing $\langle\ ,\ \rangle : M \times N \to \mathbf{Z}$ is extended to the perfect pairing $\langle\ ,\ \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \to \mathbf{R}$ of $\mathbf{R}$-vector spaces.

A subset $\sigma \subset N_{\mathbf{R}}$ is said to be a *strongly convex rational polyhedral cone* if there exists a finite subset $\{u_1, \ldots, u_s\} \subset N$ with $\sigma = \mathbf{R}_0 u_1 + \cdots + \mathbf{R}_0 u_s$ and $\sigma \cap (-\sigma) = \{0\}$, where $\mathbf{R}_0 = \{c \in \mathbf{R}\ ;\ c \geq 0\}$. A strongly convex rational polyhedral cone is denoted by a greek lowercase character as σ, τ, π, and is called simply a *cone*.

For a cone σ, we introduce the following notation. $N(\sigma)_{\mathbf{R}} = \sigma + (-\sigma)$ is the minimal linear subspace of $N_{\mathbf{R}}$ containing σ. We define $\dim \sigma = \dim_{\mathbf{R}} N(\sigma)_{\mathbf{R}}$. We set

$N(\sigma) = N \cap N(\sigma)_{\mathbf{R}}$ and $N[\sigma] = N/N(\sigma)$. We set also $M[\sigma]_{\mathbf{R}} = \sigma^{\perp} = \{x \in M_{\mathbf{R}}\ ;\ \langle x, u\rangle = 0 \text{ for all } u \in \sigma\}$, $M[\sigma] = M \cap M[\sigma]_{\mathbf{R}}$ and $M(\sigma) = M/M[\sigma]$. $M(\sigma)$ and $N(\sigma)$ are mutually dual $\mathbf{Z}$-modules of rank $\dim\sigma$, while $M[\sigma]$ and $N[\sigma]$ are mutually dual $\mathbf{Z}$-modules of rank $r - \dim\sigma$.

In this section, we fix a cone $\pi \subset N_{\mathbf{R}}$ of dimension r. Then the dual cone

$$\pi^{\vee} = \{x \in M_{\mathbf{R}}\ ;\ \langle x, u\rangle \geq 0 \text{ for all } u \in \pi\}$$

is a strongly convex rational polyhedral cone of dimension r in $M_{\mathbf{R}}$.

Since $\pi^{\vee}$ is a convex cone, $x, y \in \pi^{\vee}$ implies $x+y \in \pi^{\vee}$. Hence $M \cap \pi^{\vee}$ is a semigroup with the unit 0. Since $\pi^{\vee}$ is strongly convex, 0 is the unique invertible element of the semigroup. An element m of $M \cap \pi^{\vee}$ is said to be *irreducible* if there do not exist $m_1, m_2 \in M \cap \pi^{\vee} \setminus \{0\}$ with $m = m_1 + m_2$.

Lemma 1.1 (Gordan) *The semigroup $M \cap \pi^{\vee}$ is finitely generated. Namely, $M \cap \pi^{\vee}$ has only finite irreducible elements and the semigroup is generated by them.*

We fix a field k of any characteristic. Lemma 1.1 implies that the semigroup ring

$$S_{\pi} = k[M \cap \pi^{\vee}] = \bigoplus_{m \in M \cap \pi^{\vee}} k\mathbf{e}(m)$$

is a finitely generated k-algebra, where we define $\mathbf{e}(m)\mathbf{e}(m') = \mathbf{e}(m+m')$ for $m, m' \in M$ and $\mathbf{e}(0) = 1$. In particular, S_{π} is a Noetherian ring. For each face σ of π, we denote $\mathcal{S}(\sigma) = M \cap \pi^{\vee} \cap \sigma^{\perp}$ and $\mathcal{S}(\sigma)^{\circ} = M \cap \operatorname{rel.int}(\pi^{\vee} \cap \sigma^{\perp})$. In particular, $\mathcal{S}(\mathbf{0}) = M \cap \pi^{\vee}$. Since $\mathcal{S}(\sigma)$ is a subsemigroup of $M \cap \pi^{\vee}$, $k[\mathcal{S}(\sigma)]$ is a subring of S_{π}. Note that $k[\mathcal{S}(\sigma)]$ is also considered as the residue ring of S_{π} by the prime ideal $\bigoplus_{m \in M \cap (\pi^{\vee} \setminus \sigma^{\perp})} k\mathbf{e}(m)$. We denote by $F(\pi)$ the set of faces of π.

Lemma 1.2 *The semigroup $M \cap \pi^{\vee}$ has a decomposition*

$$M \cap \pi^{\vee} = \coprod_{\sigma \in F(\pi)} \mathcal{S}(\sigma)^{\circ}$$

as a set. If $m \in \mathcal{S}(\sigma)^{\circ}$ and $m' \in \mathcal{S}(\tau)^{\circ}$, then $m + m' \in \mathcal{S}(\rho)^{\circ}$ for $\rho = \sigma \cap \tau$.

Proof Let m be an element of $M \cap \pi^{\vee}$. Then $\sigma = \pi \cap (m = 0)$ is a face of π, where $(m = 0) = \{u \in N_{\mathbf{R}}\ ;\ \langle m, u\rangle = 0\}$. Then this is the unique face σ of π such that m is in the relative interior of $\pi^{\vee} \cap \sigma^{\perp}$. If $\tau = \pi \cap (m' = 0)$, then $\pi \cap (m + m' = 0) = \sigma \cap \tau$ since π is contained in both $(m \geq 0)$ and $(m' \geq 0)$. This implies $\rho = \sigma \cap \tau$. QED

We take a primitive element $n_0 \in N \cap \operatorname{int}\pi$. Then we can define the grading of S_{π} by $\deg \mathbf{e}(m) = \langle m, n_0\rangle$ for all m. Note that S_{π} is positively graded and $\deg \mathbf{e}(m) = 0$ if and only if $m = 0$.

For each face σ of π, we denote by $n_0(\sigma)$ the image of n_0 in $N[\sigma]$. We define the category $\operatorname{Iso}(\pi, n_0)$ as follows. Each object of $\operatorname{Iso}(\pi, n_0)$ is the semigroup $\mathcal{S}(\sigma)$ for $\sigma \in F(\pi)$ and a morphism is an isomorphism $\phi : M[\sigma] \to M[\tau]$ such that $\phi(\mathcal{S}(\sigma)) =$

$\mathcal{S}(\tau)$ and $\phi^*(n_0(\tau)) = n_0(\sigma)$. We denote this morphism also by $\phi : \mathcal{S}(\sigma) \to \mathcal{S}(\tau)$. In particular, all morphisms are isomorphisms, i.e., $\mathrm{Iso}(\pi, n_0)$ is a groupoid.

For a subgroupoid H of $\mathrm{Iso}(\pi, n_0)$, we consider the following conditions.
(1) For any element $\mathcal{S}(\sigma)$ of H, there exists a morphism $\phi : \mathcal{S}(\sigma) \to \mathcal{S}(\tau)$ in H which is not the identity map.
(2) If $\phi : \mathcal{S}(\sigma_1) \to \mathcal{S}(\sigma_2)$ is in H and it induces a non-identity map $\phi' : \mathcal{S}(\tau_1) \to \mathcal{S}(\tau_2)$ for $\tau_1, \tau_2 \in F(\pi)$ with $\sigma_1 \prec \tau_1, \sigma_2 \prec \tau_2$, then ϕ' is in H.

Let H be a subgroupoid of $\mathrm{Iso}(\pi, n_0)$ satisfying the conditions (1) and (2).

Clearly, H is decomposed into a finite union of connected components. The dimension $\dim H_0$ of a connected component is defined as $\dim H_0 = \dim \sigma$ for an object $\mathcal{S}(\sigma) \in H_0$, which is independent of the choice of the object. If $\dim H_0$ is minimal among the connected components of H, then $H' = H \setminus H_0$ is a subgroupoid satisfying the conditions (1) and (2). Actually the condition (1) is obviously satisfied for H'. The condition (2) is also satisfied since $\sigma_i \prec \tau_i$ and $\sigma_i \neq \tau_i$ imply $\dim \tau_i > \dim \sigma_i \geq \dim H_0$, and hence ϕ' is in H'.

We define the vector subspace $S_\pi(H)$ of S_π by

$$S_\pi(H) = \{ \sum_{m \in M \cap \pi^\vee} a_m \mathbf{e}(m) \; ; \; a_m = a_{\phi(m)} \text{ if } \phi : \mathcal{S}(\sigma) \to \mathcal{S}(\tau) \text{ is in } H \text{ and } m \in \mathcal{S}(\sigma) \} \; .$$

It is checked easily that this is a k-subalgebra.

The H-equivalence $\overset{H}{\sim}$ on $M \cap \pi^\vee$ is defined by $m \overset{H}{\sim} m'$ when $m = m'$ or there exists $\phi : \mathcal{S}(\sigma) \to \mathcal{S}(\tau) \in H$ with $m \in \mathcal{S}(\sigma)$ and $m' = \phi(m)$. For $m \in M \cap \pi^\vee$, we set

$$\mathbf{e}_H(m) = \sum_{m' \overset{H}{\sim} m} \mathbf{e}(m') \; .$$

Then a basis of $S_\pi(H)$ as a k-vector space is given by

$$\{\mathbf{e}_H(m) \; ; \; m \in M \cap \pi^\vee / \overset{H}{\sim}\} \; .$$

Theorem 1.3 *The ring S_π is integral over $S_\pi(H)$, and $S_\pi(H)$ is finitely generated over k.*

Proof We prove that S_π is integral over $S_\pi(H)$ by induction on the number of connected components of H. If H is empty, then $S_\pi(H) = S_\pi$ and the assertion is trivial.

Assume that H is nonempty and H_0 is a connected component of minimal dimension. Set $H' = H \setminus H_0$. Then we have the relation $S_\pi(H) \subset S_\pi(H') \subset S_\pi$. Since S_π is integral over $S_\pi(H')$ by the induction assumption, it suffices to show that $S_\pi(H')$ is integral over $S_\pi(H)$.

$S_\pi(H')$ is generated by

$$\{\mathbf{e}(m) \; ; \; m \in \mathcal{S}(\sigma)^\circ, \mathcal{S}(\sigma) \in H_0\}$$

over $S_\pi(H)$. Take an element $\mathbf{e}(m)$ in this set. Let $\{m_1, \ldots, m_d\}$ be the H-equivalent class of m with $m_1 = m$. If $\{\mathcal{S}(\sigma_1), \ldots, \mathcal{S}(\sigma_c)\}$ is the set of objects of H_0, then $e = d/c$

is an integer and each $\mathcal{S}(\sigma_i)^\circ$ contains e elements of $\{m_1, \ldots, m_d\}$. Let $p(t)$ be the monic polynomial

$$(t - \mathbf{e}(m_1)) \cdots (t - \mathbf{e}(m_d)) = t^d + u_1 t^{d-1} + \cdots + u_{d-1} t + u_d \ .$$

Then we have the equation $p(\mathbf{e}(m)) = 0$ since $m_1 = m$. The coefficient u_s is $(-1)^s$ times the elementary symmetric polynomial in $\mathbf{e}(m_1), \ldots, \mathbf{e}(m_d)$ of degree s for each $1 \le s \le d$. We will show that this symmetric polynomial is an element of $S_\pi(H)$. Let $\mathbf{e}(m_{l_1}) \cdots \mathbf{e}(m_{l_s}) = \mathbf{e}(m_{l_1} + \cdots + m_{l_s})$ be a monomial of the symmetric polynomial. If there exists no $\mathcal{S}(\sigma_i)^\circ$ which contains all $m_{l_1}, \ldots, m_{l_s}$, then $m_{l_1} + \cdots + m_{l_s}$ is contained in $\mathcal{S}(\rho)^\circ$ for ρ with $\dim \rho < \dim H_0$ by Lemma 1.2. In particular, the monomial $\mathbf{e}(m_{l_1} + \cdots + m_{l_s})$ is in $S_\pi(H)$ by the minimality of $\dim H_0$.

On the other hand, the sum f of all monomials such that $\{m_{l_1}, \ldots, m_{l_s}\}$ is contained in $\mathcal{S}(\sigma_i)^\circ$ for some i is an element of $S_\pi(H)$ since $\{\phi(m_{l_1}), \ldots, \phi(m_{l_s})\}$ has the same property for any ϕ in H_0 when this map is applicable. In order to see this fact more precisely, let X be the set of subsets of $\{m_1, \ldots, m_d\}$ consisting of s elements and contained in $\mathcal{S}(\sigma_i)^\circ$ for some i. We introduce a relation $V \sim V'$ in X when there exists $\phi : \mathcal{S}(\sigma) \to \mathcal{S}(\tau)$ in H_0 such that $V \subset \mathcal{S}(\sigma)^\circ$, $V' \subset \mathcal{S}(\tau)^\circ$ and $V' = \phi(V)$. This is an equivalence relation since H_0 is a groupoid. Let $\{X_1, \ldots, X_a\}$ be the set of equivalence classes of X. Take an arbitrary class X_q. Then for each σ_i, we set $X_{q,i} = \{V \in X_q \ ; V \subset \mathcal{S}(\sigma_i)^\circ\}$ and define an element

$$f_{q,i} = \sum_{V \in X_{q,i}} \mathbf{e}(V) \ ,$$

where $\mathbf{e}(V) = \mathbf{e}(\sum_{m \in V} m)$. Then we have $\phi(f_{q,i}) = f_{q,j}$ if $\phi : \mathcal{S}(\sigma_i) \to \mathcal{S}(\sigma_j)$ is in H_0, where we denote also by ϕ the isomorphism $k[\mathcal{S}(\sigma_i)] \to k[\mathcal{S}(\sigma_i)]$ induced by ϕ. This implies that $f_q = f_{q,1} + \cdots + f_{q,c}$ is an element of $S_\pi(H)$. Hence the sum $f = f_1 + \cdots + f_a$ is also in $S_\pi(H)$.

Thus we know that u_s is an element of $S_\pi(H)$. Since $p(\mathbf{e}(m)) = 0$, $\mathbf{e}(m)$ is integral over $S_\pi(H)$. Hence $S_\pi(H')$ is integral over $S_\pi(H)$.

Since we proved that S_π is integral over $S_\pi(H)$, the finite generation of $S_\pi(H)$ follows from that of S_π (see, for example, [AM, Proposition 7.8]). QED

2 Associated graded rings of cusp singularities

Assume $r \ge 2$. Let C be a strongly convex open cone of $N_{\mathbf{R}}$ and Γ a subgroup of $\mathrm{GL}(N)$ acting effectively on C. The quotient $D = C/\mathbf{R}_+$ has a structure of $(r-1)$-dimensional topological manifold. We assume the action of Γ on D is properly discontinuous and the quotient D/Γ is compact. It is known that there exists a Γ-invariant fan Σ of $N_{\mathbf{R}}$ such that Σ is locally finite at each point of C, the support $|\Sigma|$ is equal to $C \cup \{0\}$, the stabilizer $\Gamma_\sigma = \{\gamma \in \Gamma \ ; \gamma(\sigma) = \sigma\}$ is finite for every $\sigma \in \Sigma \setminus \{\mathbf{0}\}$ and the quotient $(\Sigma \setminus \{\mathbf{0}\})/\Gamma$ is finite. The action of Γ on D is free if and only if Γ acts on $\Sigma \setminus \{\mathbf{0}\}$ freely. In this case, the pair (C, Γ) defines a Tsuchihashi cusp singularity [T].

Lemma 2.1 *There exists a fan Σ with the following additional property. The stabilizer Γ_σ acts on the cone σ trivially for every $\sigma \in \Sigma \setminus \{\mathbf{0}\}$.*

Proof For any Γ-invariant fan, its barycentric subdivision satisfies this condition. QED

Let $C^* \subset M_{\mathbf{R}}$ be the interior of the dual cone of $\bar{C}$. For $n_0 \in N \cap C$, we define

$$\Theta^*(n_0) = \{x \in M_{\mathbf{R}} \ ; \ \langle x, \gamma(n_0)\rangle \geq 1 \text{ for all } \gamma \in \Gamma\} \ .$$

The faces of $\Theta^*(n_0)$ form a cell complex consisting of infinite number of $(r-1)$-dimensional bounded polytopes and their faces. The base space of the cell complex is the boundary $\partial\Theta^*(n_0)$ of $\Theta^*(n_0)$. For each $0 \leq i \leq r-1$, we denote by $K_i(\partial\Theta^*(n_0))$ the set of i-dimensional polytopes in the cell complex.

Lemma 2.2 *The action of Γ on $K_{r-1}(\partial\Theta^*(n_0))$ is transitive, and there exists $n_0 \in N \cap C$ such that the action is principal.*

Proof A polytope P in $K_{r-1}(\partial\Theta^*(n_0))$ is defined by $\Theta^*(n_0) \cap (\gamma(n_0) = 1)$ for an element $\gamma \in \Gamma$, which is equal to $\gamma^{-1}(\Theta^*(n_0) \cap (n_0 = 1))$. Hence the action is transitive. Since the action of Γ on C is effective, the stabilizer is trivial for a point in a non-empty open cone in C. An element $n_0 \in N \cap C$ on a rational line intersecting it makes the action principal. QED

We fix a primitive n_0 in $N \cap C$. Then $P_0 = \Theta^*(n_0) \cap (n_0 = 1)$ is an $(r-1)$-dimensional face of $\Theta^*(n_0)$. Since $\mathbf{R}_0 P_0$ is a maximal dimensional strongly convex rational polyhedral cone, there exists a cone $\pi \subset N_{\mathbf{R}}$ such that $\pi^\vee = \mathbf{R}_0 P_0$. Since $\pi^\vee \subset C^*$ and C is open, we have $\bar{C} \subset \operatorname{int}\pi \cup \{0\}$. For each $\sigma \in F(\pi)$, we denote $\sigma^* = \pi^\vee \cap \sigma^\perp$. Note that σ^* is a cone contained in $C^* \cup \{0\}$. For $\mathcal{S} = M \cap \pi^\vee$, the subcategory $H(\Gamma)$ of $\operatorname{Iso}(\pi, n_0)$ is defined as follows. An object of $H(\Gamma)$ is $\mathcal{S}(\sigma)$ such that there exist $\sigma, \tau \in F(\pi)$ and $\gamma \in \Gamma \setminus \{1\}$ satisfying $\gamma(\sigma^*) = \tau^*$, and a morphism is $\phi : M[\sigma] \to M[\tau]$ induced by γ for such σ, τ and γ.

We set $\mathcal{U} = (M \cap C^*) \cup \{0\}$. $\mathcal{U}$ is a semigroup with the unit 0, and we get the semigroup ring

$$k[\mathcal{U}] = \bigoplus_{m \in \mathcal{U}} k\mathbf{e}(m)$$

as a subring of the group ring $k[M]$. Note that $\mathcal{U}$ is not finitely generated. Actually, if $\mathcal{U}$ was finitely generated, it generates a strongly convex closed cone D contained in $C^* \cup \{0\}$. Since $C^* \setminus D$ is a nonempty open cone, it contains a point m in M. This is a contradiction, since $m \in \mathcal{U} \subset D$. In particular, $k[\mathcal{U}]$ is not finitely generated over k. $k[\mathcal{U}]$ has a structure of graded ring by defining $\deg \mathbf{e}(m) = \langle m, n_0\rangle$. Namely, $k[\mathcal{U}]_d$ is the k-vector space with the basis $\{\mathbf{e}(m) \ ; \ m \in \mathcal{U}, \langle m, n_0\rangle = d\}$.

Since $\pi^\vee = \mathbf{R}_0 P_0 \subset C^* \cup \{0\}$, $S_\pi = k[M \cap \pi^\vee]$ is a subalgebra of $k[\mathcal{U}]$.

Since n_0 is in the open cone C, the convex set $C^* \cap (n_0 < d)$ is bounded for every $d > 0$, where $(n_0 < d)$ is the open half space $\{x \in M_{\mathbf{R}} \ ; \ \langle x, n_0\rangle < d\}$. In particular, $M \cap C^* \cap (n_0 < d)$ is a finite set.

For each positive integer d, we define the ideal $I_d \subset k[\mathcal{U}]$ by

$$I_d = (\mathbf{e}(m) \ ; \ m \in \mathcal{U}, \langle m, n_0 \rangle \geq d) \ .$$

Note that $\{\mathbf{e}(m) \ ; \ \langle m, n_0 \rangle \geq d\}$ is a k-basis of the ideal I_d. Since the complement of this set in $\mathcal{U}$ is finite, the quotient $k[\mathcal{U}]/I_d$ is an Artinian ring. We consider the completion $k[[\mathcal{U}]] = \operatorname{proj\,lim}_i k[\mathcal{U}]/I_i$. It is easy to see that this k-algebra is written as the product

$$k[[\mathcal{U}]] = \prod_{m \in \mathcal{U}} k\mathbf{e}(m) \ .$$

In particular, it does not depend on the choice of n_0.

An element of $k[[\mathcal{U}]]$ is written as an infinite sum $f = \sum_{m \in \mathcal{U}} a_m \mathbf{e}(m)$. The action of Γ on $k[[\mathcal{U}]]$ is defined by

$$\gamma(f) = \sum_{m \in \mathcal{U}} a_m \mathbf{e}(\gamma(m))$$

for $f \in k[[\mathcal{U}]]$ and $\gamma \in \Gamma$. Note that $(\gamma\gamma')(f) = \gamma'(\gamma(f))$. The ring $k[[\mathcal{U}]]$ has an induced filtration $\{\hat{I}_d\}$ of ideals, where $\hat{I}_d = k[[\mathcal{U}]]I_d$ for each $d \geq 0$. Note that

$$\hat{I}_d/\hat{I}_{d+1} = I_d/I_{d+1} = k[\mathcal{U}]_d \ .$$

Hence the associated graded ring

$$G(k[[\mathcal{U}]]) = \bigoplus_{d=0}^{\infty} \hat{I}_d/\hat{I}_{d+1}$$

of $k[[\mathcal{U}]]$ for this filtration is naturally isomorphic to $k[\mathcal{U}]$.

For each $f \in k[[\mathcal{U}]] \setminus \{0\}$, we denote by $L(f)$ the image of f in $\hat{I}_d/\hat{I}_{d+1}$ for the maximal d with $f \in \hat{I}_d$ and call it the *leading part* of f. Note that $f - L(f)$ is in $\hat{I}_{d+1}$.

We consider the invariant subring $A = k[[\mathcal{U}]]^\Gamma$. We can show that this is the completion of the local ring of the cusp singularity when $k = \mathbf{C}$ and the action of Γ on C is free. The ring A has the filtration defined by $I(A)_d = \hat{I}_d \cap A$ for $d \geq 0$. We define the associated graded ring B by

$$B = G(A) = \bigoplus_{d=0}^{\infty} I(A)_d/I(A)_{d+1} \ .$$

Theorem 2.3 *B is equal to $S_\pi(H(\Gamma))$ in $k[\mathcal{U}]$. In particular, B is finitely generated over k.*

Proof Let $f = \sum_{m \in \mathcal{U}} a_m \mathbf{e}(m)$ be an element of $I(A)_d$. If $a_m \neq 0$, then $a_{\gamma(m)} = a_m \neq 0$ and $\langle m, \gamma(n_0) \rangle = \langle \gamma(m), n_0 \rangle \geq d$ for every $\gamma \in \Gamma$. This implies m is in $d\Theta(n_0)$ since $\Theta(n_0) = \{x \in C^* \ ; \ \langle x, \gamma(n_0) \rangle \geq 1 \text{ for all } \gamma \in \Gamma\}$. If $\deg \mathbf{e}(m) = d$, then since $d\Theta(n_0) \cap (n_0 = d) = dP_0$, $\mathbf{e}(m)$ is in $k[M \cap \pi^\vee]_d$. Since $L(f)$ is the sum of $a_m \mathbf{e}(m)$ with $\deg \mathbf{e}(m) = d$, we have $L(f) \in k[M \cap \pi^\vee]_d$. If $\phi : \mathcal{S}(\sigma) \to \mathcal{S}(\tau)$ is in $H(\Gamma)$, $m \in \mathcal{S}(\sigma)^\circ$ and $\mathbf{e}(m) \in k[\mathcal{S}(\sigma)]_d$, then there exists $\gamma \in \Gamma$ such that $\phi(m) = \gamma(m)$. Since f is

Γ-invariant and $n_0(\sigma) = n_0(\tau)$, we have $n_0(\phi(m)) = n_0(m) = d$ and $a_{\phi(m)} = a_m$. Hence $L(f)$ is in $S_\pi(H(\Gamma))$. Thus we know that B is a subring of $S_\pi(H(\Gamma))$.

In order to show $B = S_\pi(H(\Gamma))$, it suffices to see that $\mathbf{e}_{H(\Gamma)}(m)$ is in B for every $m \in M \cap \pi^\vee$. This is clear since $\mathbf{e}_{H(\Gamma)}(m) = L(f)$ for $f = \sum_{\gamma\in\Gamma/\mathrm{St}(m)} \mathbf{e}(\gamma(m)) \in A$, where $\mathrm{St}(m)$ is the stabilizer of m and $\Gamma/\mathrm{St}(m)$ is the set of the right cosets. QED

Let $k[[x_1, \dots, x_s]]$ be the power series ring of s variables. The local ring $k[[x_1, \dots, x_s]]$ has the filtration $\{(x_1, \dots, x_s)^d\}$ defined by the maximal ideal $(x_1, \dots, x_s)$. For any given elements $u_1, \dots, u_s$ of I_1, there exists a continuous k-algebra homomorphism $\phi : k[[x_1, \dots, x_s]] \to k[[\mathcal{U}]]$ such that $\phi(x_i) = u_i$ for every i. Actually, an element $f \in k[[x_1, \dots, x_s]]$ is written as

$$f = f_0 + f_1 + f_2 + \cdots$$

with f_i a homogeneous polynomial of degree i for every i. Then $f_i(u_1, \dots, u_s) \in I_i$ for every i, and

$$f_0(u_1, \dots, u_s) + f_1(u_1, \dots, u_s) + f_2(u_1, \dots, u_s) + \cdots$$

defines an element of $k[[\mathcal{U}]]$ which should be $\phi(f)$. If $u_1, \dots, u_s$ are in the subring A, the image of ϕ is in A since every finite sum $\sum_{i=0}^{d} f_i(u_1, \dots, u_s)$ is in A and A is closed (cf. Proposition 3.2).

It is known that formal power series ring with finite variables over a Noetherian ring is Noetherian [CC, 18]. We have the following theroem.

Theorem 2.4 (cf. [CC, 18, Théorèm 4]) *Let $\{y_1, \dots, y_s\}$ be a set of homogeneous elements of B which generates B over k. Assume $y_i \in I(A)_{l_i}/I(A)_{l_i+1}$ and y_i is represented by $u_i \in I(A)_{l_i} \setminus I(A)_{l_i+1}$ for each i. Then the homorphism $\phi : k[[x_1, \dots, x_s]] \to A$ defined by $u_1, \dots, u_s$ is surjective. In particular, A is a Noetherian local ring.*

Proof Let w be an arbitrary element of A. We define inductively a sequence of polynomials $f_i \in k[x_1, \dots, x_s]$ such that $f_i(u_1, \dots, u_s) \in I(A)_i$ for every i,

$$w - \{f_0(u_1, \dots, u_s) + f_1(u_1, \dots, u_s) + \cdots + f_d(u_1, \dots, u_s)\}$$

is in $I(A)_{d+1}$ for every d and $\{f_0 + f_1 + \cdots + f_d\}$ is a Cauchy sequence. Assume $f_0, \dots, f_{d-1}$ are defined so that they satisfy the first two conditions. Then

$$g = w - \{f_0(u_1, \dots, u_s) + f_1(u_1, \dots, u_s) + \cdots + f_{d-1}(u_1, \dots, u_s)\}$$

is an element of $I(A)_d$. The image $\bar{g}$ of g in $I(A)_d/I(A)_{d+1}$ is represented by a linear combination

$$\sum_{e_1,\dots,e_s} a_{e_1,\dots,e_s} y_1^{e_1} \cdots y_s^{e_s}$$

with coefficients in k, where the summation is taken over $e_1, \dots, e_s \geq 0$ satisfying $e_1 l_1 + \cdots + e_s l_s = d$. Since this element is represented by $f_d(u_1, \dots, u_s)$ for the polynomial

$$f_d(x_1, \dots, x_s) = \sum_{e_1,\dots,e_s} a_{e_1,\dots,e_s} x_1^{e_1} \cdots x_s^{e_s} \ ,$$

we have

$$\begin{aligned} & w - \{f_0(u_1, \ldots, u_s) + f_1(u_1, \ldots, u_s) + \cdots + f_d(u_1, \ldots, u_s)\} \\ = \; & g - f_d(u_1, \ldots, u_s) \in I(A)_{d+1} \; . \end{aligned}$$

By the equality $e_1 l_1 + \cdots + e_s l_s = d$, the degree $e_1 + \cdots + e_s$ of the monomial $x_1^{e_1} \cdots x_s^{e_s}$ of f_d is at least $d/\max\{l_1, \ldots, l_s\}$. This implies that $\{f_0 + f_1 + \cdots + f_d\}$ is a Cauchy sequence. Hence the infinite sum $f = f_0 + f_1 + \cdots$ defines an element of $k[[x_1, \ldots, x_s]]$ and $\phi(f) = w$. Since w is arbitrary, ϕ is surjective.

Since the power series ring is Noetherian, so is the homomorphism image A. QED

3 Some results related to Gröbner basis

We think k has a discrete topology, and the k-algebra $k[[\mathcal{U}]] = \prod_{m \in \mathcal{U}} k\mathbf{e}(m)$ has the product topology. This topology is equal to that of the completion of $k[\mathcal{U}]$ with respect to the filtration $\{I_d\}$.

For a subset E of $k[[\mathcal{U}]]$, an element $x = \sum_{m \in \mathcal{U}} a_m \mathbf{e}(m)$ is in the closure of E if and only if there exists a sequence $\{y_i\}$ of elements of E such that the image of y_d to $k[[\mathcal{U}]]/I_d$ is equal to that of x for every d. This implies that a vector subspace $V \subset k[[\mathcal{U}]]$ is closed if and only if

$$V = \bigcap_{d=0}^{\infty} (V + I_d) \; .$$

For a vector subspace $V \subset k[[\mathcal{U}]]$, we denote

$$L(V) = \bigoplus_{d=0}^{\infty} (V \cap I_d + I_{d+1})/I_{d+1} \; .$$

This is a vector subspace of $k[\mathcal{U}]$.

Lemma 3.1 *Let V_1, V_2 be vector subspaces of $k[[\mathcal{U}]]$ with $V_1 \subset V_2$. Assume that V_1 is closed. Then $V_1 = V_2$ if and only if $L(V_1) = L(V_2)$.*

Proof It is obvious that $L(V_1) = L(V_2)$ if $V_1 = V_2$. Hence it suffices to show that any element $x \in V_2$ is in V_1 if $L(V_1) = L(V_2)$. For any $d \geq 0$, we have

$$\begin{aligned} & \dim_k (V_1 + I_d)/I_d \\ = \; & \sum_{i=0}^{d-1} \dim_k (V_1 \cap I_i + I_{i+1})/I_{i+1} \\ = \; & \sum_{i=0}^{d-1} \dim_k (V_2 \cap I_i + I_{i+1})/I_{i+1} \\ = \; & \dim_k (V_2 + I_d)/I_d \; . \end{aligned}$$

Since $(V_1 + I_d)/I_d \subset (V_2 + I_d)/I_d$, these vector spaces are equal. Hence x is in $V_1 + I_d$ for every d. This implies $x \in V_1$ since V_1 is closed. QED

The following proposition was used in Section 2.

Proposition 3.2 *The ring $A \subset k[[\mathcal{U}]]$ is closed.*

Proof Let $x = \sum_{m \in \mathcal{U}} a_m \mathbf{e}(m)$ be an element of the closure of A. It suffices to show that x is in A. For $m \in \mathcal{U}$ and $\gamma \in \Gamma$, let $d = \max\{\langle m, n_0 \rangle, \langle \gamma(m), n_0 \rangle\} + 1$. Since $x \in A + I_d$, there exists $y \in A$ with $x - y \in I_d$. This implies $a_m = a_{\gamma(m)}$ since the coefficients of x and y are equal for every $\mathbf{e}(m')$ with $\langle m', n_0 \rangle \leq d$. Since m and γ are arbitrary, x is an element of A. QED

Theorem 3.3 *Any ideal of A is closed in $k[[\mathcal{U}]]$. In particular, ideals I_1, I_2 of A with $I_1 \subset I_2$ are equal if and only if $L(I_1) = L(I_2)$.*

Proof Let I be an ideal of A. We may assume $I \neq A$. Since A is a complete Noetherian local ring by Theorem 2.4, all ideals are closed in $I(A)_1$-adic topology by Artin-Rees's lemma. Since the induced topology of A is $I(A)_1$-admissible, it is equal to the $I(A)_1$-adic topology (see EGA I, Chap. 0, §7). Hence any ideal I of A is closed in $k[[\mathcal{U}]]$. The second assertion follows from Lemma 3.1 since ideals are closed. QED

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